

Self-Gravitational Correction of the “Vacuum Polarization” Feynman Diagram Using Full Einstein Equation Propagation of the Intermediate Virtual Gravitons

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Abstract

The second-order “vacuum polarization” radiative correction insertion Feynman diagram is an ultraviolet divergent virtual object whose Fourier transform to configuration space-time has the properties of a virtual stress-energy tensor. Though well-defined, this virtual stress-energy is non-integrably singular on the light cone—which produces the aforementioned ultraviolet divergence of its four-momentum space Fourier transform. To properly calculate the self-gravitationally corrected version of this singular virtual stress-energy, the usual graviton-exchange ladder sum approximations are eschewed in favor of full, self-consistent Einstein equation propagation of the intermediate virtual gravitons, which takes into account their important non-linear interactions with each other. (As a by-product, the subsequent perturbative treatment of these non-linearities is avoided, which eliminates the source of the ultraviolet divergences of the second-quantized gravity theory itself.) The resulting corrected virtual stress-energy is non-singular everywhere and Fourier-transforms convergently to a finite corrected version of the diagram. This corrected diagram makes no contribution to charge renormalization (as could be expected of a diagram involving but a single transient virtual pair), and its dynamical behaviour accords with the standard quantum electrodynamics result except at inaccessibly extreme values of the momentum transfer, $|q^2| \gtrsim (G\epsilon^2)^{-1}$. There, the standard logarithmic rise with momentum transfer which this diagram contributes to the effective coupling strength falls away, as the diagram proceeds instead to decrease strongly toward zero. The same self-gravitational correction is made to the closely related quartically divergent second-order vacuum-to-vacuum amplitude correction Feynman diagram, and it is found that the result vanishes identically.

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I Introduction

Self-gravitational correction as a possible physical mechanism for resolving ultraviolet divergences in quantum field theories has been discussed over the years by a number of authors. Some of the earliest published speculations along this line were put forth by Landau [1], Klein [2, 3, 4], Pauli [5], and Deser [6]. A definitive determination of the finite mass of the classical point charge was worked out by Arnowitt, Deser, and Misner [7] in the context of a non-singular metric (different from the conventional singular Reissner–Nordström metric for this situation). This mass for the classical point charge has recently been independently rederived by the author [8]. Ref. [7] emphasizes that, for this highly singular source of gravitational field, the countervailing (negative) effective energy density due to the resulting very strong gravitational field itself, carefully taken into account in self-consistent fashion, is critical to obtaining the correct finite value for the point charge’s mass. Subsequent work by DeWitt [9], Khriplovich [10], Salam and Strathdee [11], and Isham, Salam, and Strathdee [12] has approached the ultraviolet divergences of quantum electrodynamics through correcting the lowest-order Feynman diagrams in which they occur by various types of summed graviton-exchange ladders. These simple graviton-exchange processes do not, however, include the non-linear feature that the virtual gravitons interact *with each other*, which was found in Ref. [7] to be so crucial to the proper self-gravitational resolution of the infinite mass of the classical point charge. Moreover, the graviton-exchange ladder sums are generally not by themselves gravitationally gauge-invariant [9, 10], and can in addition introduce difficulties with electromagnetic gauge invariance [12]. Khriplovich looked into the effects of making different choices of gravitational gauge in his approach, and found that some of these seemed unlikely to result in the suppression of the ultraviolet divergences [13]. Furthermore, the quantized electron mass electromagnetic correction obtained in Ref. [8] (by straightforward extension of the non-singular metric approach that was used for the classical point charge) is found to incorporate a gravitationally induced effective cutoff radius which depends in a physically sensible way on the electron charge—in contrast with the puzzling charge independence of that effective cutoff radius in the result obtained by means of the ladder-sum approaches of Refs. [10] and [12].

The self-gravitational treatment in this paper of the ultraviolet-divergent second-order “vacuum polarization” radiative correction insertion Feynman diagram of Fig. 1 is essentially the same as the quantum field theoretic approach of Ref. [12] in most respects. Since the focus here is on self-gravitational suppression of the quantum electrodynamically induced ultraviolet divergence, the Ref. [12] restriction that gravity is only coupled to the electromagnetic interaction term of the action integral is observed, as it is this interaction term which generates the Feynman diagram electromagnetic vertices, without which there could be no electromagnetically induced ultraviolet divergences. Thus the diagram of Fig. 1 is permitted to couple to intermediate virtual gravitons which travel between its two vertices—these produce the part of its self-gravitational correction which could be capable of impinging on its ultraviolet divergence. Ref. [12] uses the “graviton superpropagator” approximation framework to describe the propagation of these intermediate virtual gravitons. This “graviton superpropagator” is, of course, a member of the species of summed graviton-exchange ladders—whose shortcomings in describing ultrastrong gravitational effects engendered by an ultraviolet divergence were touched upon in the discussion above. The “superpropagator” approximation of Ref. [12] is therefore replaced by use of the full Einstein field equation to self-consistently describe the propagation of the intermediate virtual gravitons. Fourier transformed from four-momentum space to configuration space-time, the insertion diagram of Fig. 1 has the proper dimensions and other attributes of a stress-energy tensor, although, being complex-valued, it

is, of course, a virtual stress-energy. It also possesses a non-integrable singularity on the light cone, which is the feature that results in its ultraviolet divergence upon Fourier transformation back to four-momentum space. This complex-valued virtual stress-energy gives rise, via the Einstein equation, to a complex-valued virtual metric that far more satisfactorily describes the intermediate virtual graviton propagation in this ultraviolet-divergent situation than does the “superpropagator” approximation of Ref. [12]. From this virtual metric we obtain the self-gravitationally corrected virtual stress-energy by the method of Ref. [14], namely we calculate $-(8\pi G)^{-1}$ times the part of the Einstein tensor which is linear in the virtual gravitational field (that field is defined as the virtual metric tensor minus the Minkowskian flat-space metric tensor). This self-gravitationally corrected virtual stress-energy tensor turns out to be non-singular (it actually vanishes on the light cone), and it Fourier transforms back to four-momentum space as the self-gravitationally corrected and ultraviolet-convergent version of the insertion Feynman diagram of Fig. 1. A notable by-product of full Einstein equation propagation of virtual gravitons, which takes proper account of their non-linear interactions with each other, is that the subsequent perturbative treatment of these non-linearities is thus avoided—eliminating the source of the ultraviolet divergences of the second-quantized gravity theory itself.

It is worthwhile to point out here that a systematic general approach to the full theory of self-gravitationally coupled quantum electrodynamics along the lines discussed above would involve the usual perturbation expansion in the purely electromagnetic interaction combined with a stationary treatment of the gravitational field degrees of freedom in the Feynman path integration of the total action. The virtual stress-energy occurring in the resulting Einstein equation for the virtual metric could then be simplified by eliminating terms which do *not* contribute in an ultraviolet-divergent fashion to the problem being treated, since any such “ordinary” self-gravitational corrections to quantum electrodynamics can be expected to be completely negligible at attainable energies.¹ The justification of the key stationary approximation for the gravitational degrees of freedom in the Feynman path integration can be argued as follows. When gravitational effects are sufficiently weak, gravity theory is essentially linear, and for linear field theories the stationary approximation to Feynman path integration in their field variables happens to be exact. Moreover, the gravitational action contains terms such as the integral over space-time of the curvature scalar *divided* by the factor $16\pi G$, which become *large* compared to \hbar long before the non-linear corrections to gravity theory, which are typically suppressed by a factor of G relative to the linear terms, begin to contribute significantly. Of course, the “classic” (i.e., Correspondence Principle) situation which justifies the stationary approximation to Feynman path integration occurs precisely when the action is large compared to \hbar . The very considerable *overlap* here between this state of affairs (which always obtains for sufficiently *strong* gravity) and the “effectively linear theory” justification for the stationary approximation (up to the point where the non-linear effects of gravitation intervene) ensures that the stationary approximation is *uniformly* valid for any strength of gravitational effect which may be present. It seems plausible that the general approach envisioned here will essentially parallel that of Ref. [12], except that the “graviton superpropagator” of the latter will be replaced by full Einstein equation propagation of virtual intermediate gravitons, and perturbative treatment of non-linear parts of the Einstein equation cannot arise. Detailed formal development of this envisioned systematic general approach to self-gravitationally coupled quantum electrodynamics is not attempted in the present paper; that very substantial endeavor is deferred to a future time.

The organization of this paper is as follows. In Section II the usual four-momentum space

¹However, see Ref. [15] for the situation at energies far above the Planck scale, where the *entire* dynamics turns out to reduce to the *full* effect of just classical gravitation coupled to classical particle motion.

treatment of the insertion diagram of Fig. 1. is reviewed. In Section III the full details of the analogous treatment in configuration space–time, where this diagram is seen to be a virtual stress-energy tensor, are spelled out (the virtual stress-energy character of the higher-order versions of this diagram is demonstrated as well). In Section IV this virtual stress-energy is propagated with the full Einstein equation to obtain its corresponding virtual metric, which in turn yields the self-gravitationally corrected virtual stress-energy. The consequences for the four-momentum space diagram itself of this self-gravitational correction, such as the suppression of the ultraviolet divergence, the absence of any contribution to charge renormalization, and its behaviour at extreme momentum transfer, are set forth in Section V. There it is also pointed out that the closely related second-order electromagnetic vacuum-to-vacuum amplitude correction diagram of Fig. 2, which has an extreme (quartic) ultraviolet divergence, can be shown to vanish identically after the same self-gravitational correction. This obviates the need for the usual fiat injunction that such a “disconnected” diagram is to be “discarded”, notwithstanding its strongly infinite value.

II The “vacuum polarization” insertion in momentum space

In this Section the standard four-momentum space approach to the ultraviolet-divergent “vacuum polarization” radiative correction insertion diagram shown in Fig. 1 is reviewed. That diagram consists of a single photon propagator, one end of which is attached to one of the two vertices of a virtual electron–positron pair loop. This ultraviolet-divergent insertion may be added between any vertex and its attached photon line in any quantum electrodynamics Feynman diagram, in order to generate one of that diagram’s radiative corrections. The second-order “vacuum polarization” radiative correction insertion of Fig. 1 is expressed as [16]:

$$\hat{T}^{\mu\nu}(q) = \frac{-e^2}{q^2 + i\epsilon} \hat{L}^{\mu\nu}(q), \quad (1a)$$

where the electron–positron virtual pair loop portion is

$$\hat{L}^{\mu\nu}(q) \equiv -4\pi i \int \frac{d^4p}{(2\pi)^4} \text{Tr} \left(\gamma^\mu \hat{S}_F(p) \gamma^\nu \hat{S}_F(p - q) \right), \quad (1b)$$

with the spinor propagator $\hat{S}_F(p)$ given by

$$\hat{S}_F(p) = \frac{\not{p} + m}{p^2 - m^2 + i\epsilon}. \quad (1c)$$

From Eqs. (1b) and (1c) above, we can see that the loop factor $\hat{L}^{\mu\nu}(q)$ is apparently quadratically divergent, and that the entire insertion, $\hat{T}^{\mu\nu}(q)$, is dimensionless, which an insertion necessarily must be. If the photon line which one may attach to the loop portion of this insertion happens to be an external one with $q^2 = 0$, then gauge invariance with respect to permissible variation of this external photon’s polarization four-vector implies that we must have

$$\hat{L}^{\mu\nu}(q) q_\nu = 0 \quad \text{for } q^2 = 0. \quad (2)$$

It turns out that $\hat{L}^{\mu\nu}(q)$ may be reexpressed in the form [17],

$$\hat{L}^{\mu\nu}(q) = \left(\eta^{\mu\nu} q^2 - q^\mu q^\nu \right) \hat{h}(q^2) + \eta^{\mu\nu} \hat{d}(q^2),$$

where we use $\eta^{\mu\nu}$ to denote the flat-space (Minkowskian) metric tensor. The term proportional to $\hat{h}(q^2)$ above automatically satisfies the gauge-invariance condition of Eq. (2). In Ref. [17], an argument is made that the object $\hat{d}(q^2)$ above, although appearing to be quadratically divergent, in fact vanishes. Thus we may write

$$\hat{L}^{\mu\nu}(q) = (\eta^{\mu\nu} q^2 - q^\mu q^\nu) \hat{h}(q^2) \quad (3a)$$

and, from Eq. (1a),

$$\hat{T}^{\mu\nu}(q) = \frac{-e^2}{q^2 + i\epsilon} (\eta^{\mu\nu} q^2 - q^\mu q^\nu) \hat{h}(q^2), \quad (3b)$$

where $\hat{h}(q^2)$ is only logarithmically divergent [17]. We now see that the insertion $\hat{T}^{\mu\nu}(q)$ is symmetric in its two Lorentz indices μ and ν , dimensionless, and always satisfies

$$q_\mu \hat{T}^{\mu\nu}(q) = 0. \quad (4)$$

Thus its configuration space–time Fourier transform $T^{\mu\nu}(x)$ must also be symmetric in its two Lorentz indices, have vanishing four-divergence, and have the dimensions of q^4 , which are those of energy density.² So $T^{\mu\nu}(x)$ has the attributes of a stress-energy tensor, albeit a virtual one—we shall see in Section III that it is complex-valued.

Before entering the detailed calculation of $T^{\mu\nu}(x)$, which we shall undertake in the next section, we first need to show how the logarithmic divergences of $\hat{L}^{\mu\nu}(q)$, $\hat{T}^{\mu\nu}(q)$ and $\hat{h}(q^2)$ in Eqs. (3) are customarily handled. One splits $\hat{h}(q^2)$ into a logarithmically divergent part which is independent of q^2 , and a q^2 -dependent part which is convergent:

$$\hat{L}^{\mu\nu}(q) = (\eta^{\mu\nu} q^2 - q^\mu q^\nu) \left[\hat{h}|_{q^2=0} + (\hat{h}(q^2) - \hat{h}|_{q^2=0}) \right] \quad (5a)$$

or

$$\hat{T}^{\mu\nu}(q) = -\frac{e^2}{q^2} (\eta^{\mu\nu} q^2 - q^\mu q^\nu) \left[\hat{h}|_{q^2=0} + (\hat{h}(q^2) - \hat{h}|_{q^2=0}) \right]. \quad (5b)$$

The logarithmically divergent $\hat{h}|_{q^2=0}$ is absorbed into charge renormalization,³ while the observable q -dependent dynamical effects of this insertion $\hat{T}^{\mu\nu}(q)$ are wholly attributed to the convergent object $(\hat{h}(q^2) - \hat{h}|_{q^2=0})$. As well as being convergent, $(\hat{h}(q^2) - \hat{h}|_{q^2=0})$ is also proportional to q^2 for sufficiently small $|q^2|$, i.e., for $|q^2| \ll m^2$, where m is the electron mass. For the purpose of passing to configuration space–time, which is where the self-gravitational correction of $T^{\mu\nu}(x)$ must be worked out using the Einstein equation, we rewrite Eq. (5b) in the form

$$\hat{T}^{\mu\nu}(q) = e^2 (\eta^{\mu\nu} q^2 - q^\mu q^\nu) \hat{H}(q^2) - e^2 \left(\eta^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right) \hat{h}|_{q^2=0}, \quad (6)$$

where

$$\hat{H}(q^2) \equiv -\frac{1}{q^2} (\hat{h}(q^2) - \hat{h}|_{q^2=0}). \quad (7)$$

The first term of Eq. (6) is clearly convergent, while the logarithmically divergent second term will not contribute to the Fourier transform $T^{\mu\nu}(x)$ for any $x^\mu \neq 0$. Thus, from Eqs. (6) and (7), the configuration space–time Fourier transforms $T^{\mu\nu}(x)$ and $H(x^2)$ are infinity-free for all $x^\mu \neq 0$.

²We use the conventional system of units where $\hbar = c = 1$.

³It does seem questionable to thus imbue an *ultraviolet* divergence with a fundamental rôle in determining the scale of the extreme *low-energy* scattering strength via such an effective modification of electronic charge, even leaving aside the puzzle of how a diagram which involves but a single transient virtual pair can manage to renormalize charge by effectively polarizing the entire vacuum.

III The “vacuum polarization” insertion in configuration space

In Eqs. (1)–(5) of Section II, the usual four-momentum space treatment of the “vacuum polarization” insertion was outlined. The detailed analogous treatment in configuration space–time is now carried out in order to obtain the explicit form of the virtual stress-energy tensor $T^{\mu\nu}(x)$, whose self-gravitational correction $T_G^{\mu\nu}(x)$ one subsequently wishes to calculate.

The goal here is to Fourier-transform the $\hat{T}^{\mu\nu}(q)$ of Eq. (1a) to configuration space–time, but the initial effort will concentrate on the Fourier transform of just its loop factor $\hat{L}^{\mu\nu}(q)$. Using Eq. (1b) we obtain

$$\begin{aligned}
L^{\mu\nu}(x) &= -4\pi i \operatorname{Tr} \left[\int \frac{d^4 q}{(2\pi)^4} e^{-iq \cdot x} \int \frac{d^4 p}{(2\pi)^4} \gamma^\mu \hat{S}_F(p) \gamma^\nu \hat{S}_F(p - q) \right] \\
&= -4\pi i \operatorname{Tr} \left[\int \frac{d^4 q}{(2\pi)^4} e^{-iq \cdot x} \int \frac{d^4 p}{(2\pi)^4} \gamma^\mu \right. \\
&\quad \times \left. \int d^4 x_1 e^{ip \cdot x_1} S_F(x_1) \gamma^\nu \int d^4 x_2 e^{i(p-q) \cdot x_2} S_F(x_2) \right] \\
&= -4\pi i \operatorname{Tr} \left[\int d^4 x_1 \int d^4 x_2 \gamma^\mu S_F(x_1) \gamma^\nu S_F(x_2) \delta^{(4)}(x + x_2) \delta^{(4)}(x_1 + x_2) \right] \\
&= -4\pi i \operatorname{Tr} [\gamma^\mu S_F(x) \gamma^\nu S_F(-x)], \tag{8}
\end{aligned}$$

where, using Eq. (1c),

$$\begin{aligned}
S_F(x) &= \int \frac{d^4 p}{(2\pi)^4} e^{-ip \cdot x} \hat{S}_F(p) \\
&= \int \frac{d^4 p}{(2\pi)^4} e^{-ip \cdot x} \left(\frac{\not{p} + m}{p^2 - m^2 + i\epsilon} \right) \\
&= (i\not{\partial} + m) \int \frac{d^4 p}{(2\pi)^4} \frac{e^{-ip \cdot x}}{p^2 - m^2 + i\epsilon} \\
&= (i\not{\partial} + m) \Delta_F(x^2). \tag{9}
\end{aligned}$$

It is convenient to have a manifestly scalar representation of the function $\Delta_F(x^2)$. With some of the details of the variable changes and intermediate Gaussian integration steps left to the reader, its derivation is as follows:

$$\begin{aligned}
\Delta_F(x^2) &= \int \frac{d^4 p}{(2\pi)^4} \frac{e^{-ip \cdot x}}{p^2 - m^2 + i\epsilon} \\
&= -i \int_0^\infty ds \int \frac{d^4 p}{(2\pi)^4} e^{-ip \cdot x} e^{i(p^2 - m^2 + i\epsilon)s} \\
&= -\frac{1}{16\pi^2} \int_0^\infty \frac{ds}{s^2} e^{-ix^2/(4s)} e^{-i(m^2 - i\epsilon)s} \\
&= -\frac{m^2}{4\pi^2} \int_0^\infty d\kappa e^{-i\kappa m^2 x^2} e^{-i(1-i\epsilon)/(4\kappa)}. \tag{10}
\end{aligned}$$

At $x^2 = 0$, the above integral representation of $\Delta_F(x^2)$ diverges, so a δ -parametrized damping factor is inserted:

$$\Delta_F(x^2) = \lim_{\epsilon, \delta \rightarrow 0+} -\frac{m^2}{4\pi^2} \int_0^\infty d\kappa e^{-i\kappa(m^2 x^2 - i\delta)} e^{-i(1-i\epsilon)/(4\kappa)}. \tag{11}$$

Near the light cone, $\Delta_F(x^2)$ has the asymptotic form

$$\Delta_F(x^2) \sim \lim_{\delta \rightarrow 0^+} \frac{i}{4\pi^2} \left(\frac{1}{x^2 - i\delta} \right) \quad \text{as } x^2 \rightarrow 0 \quad (12a)$$

or

$$\Delta_F(x^2) \sim \frac{i}{4\pi^2} \mathcal{P} \frac{1}{x^2} - \frac{1}{4\pi} \delta(x^2) \quad \text{as } x^2 \rightarrow 0. \quad (12b)$$

With $\Delta_F(x^2)$ now in the desired representation given by Eq. (11), we return our attention to the loop tensor $L^{\mu\nu}(x)$ of Eq. (8), insert Eq. (9) into it, and carry out the trace using the trace theorems of Ref. [18]:

$$\begin{aligned} L^{\mu\nu}(x) &= -4\pi i \operatorname{Tr}[\gamma^\mu S_F(x) \gamma^\nu S_F(-x)] \\ &= -4\pi i \operatorname{Tr} \left[\gamma^\mu \left((i\not{\partial} + m) \Delta_F(x^2) \right) \gamma^\nu \left((-i\not{\partial} + m) \Delta_F(x^2) \right) \right] \\ &= -16\pi i \left[2(\partial^\mu \Delta_F)(\partial^\nu \Delta_F) - \eta^{\mu\nu} (\partial_\alpha \Delta_F)(\partial^\alpha \Delta_F) + \eta^{\mu\nu} m^2 (\Delta_F)^2 \right], \end{aligned} \quad (13)$$

where $\eta^{\mu\nu}$ is the flat-space (Minkowskian) metric tensor. We see that $L^{\mu\nu}(x)$ is symmetric in its two Lorentz indices. If we take the four-divergence of $L^{\mu\nu}(x)$, we obtain

$$\begin{aligned} \partial_\nu L^{\mu\nu}(x) &= -16\pi i \left[2(\partial^\mu \Delta_F)(\partial_\nu \partial^\nu \Delta_F) + 2m^2(\partial^\mu \Delta_F) \Delta_F \right] \\ &= -16\pi i \left[2(\partial^\mu \Delta_F) \left((\partial_\nu \partial^\nu + m^2) \Delta_F \right) \right] \\ &= -16\pi i \left[2 \left(\partial^\mu \Delta_F(x^2) \right) \left(-\delta^{(4)}(x) \right) \right] \\ &= -16\pi i \left[-4x^\mu \Delta'_F(x^2) \delta^{(4)}(x) \right], \end{aligned} \quad (14)$$

where we have used the fact that the basic definition of Δ_F (see Eq. (9)) implies that it is a Green function for the Klein–Gordon equation: $(\partial_\nu \partial^\nu + m^2) \Delta_F = -\delta^{(4)}(x)$. We see from Eq. (14) that

$$\partial_\nu L^{\mu\nu}(x) = 0 \quad \text{if } x \neq 0. \quad (15)$$

Also, from Eq. (11) we obtain

$$\Delta'_F(x^2) = \frac{i(m^2)^2}{4\pi^2} \int_0^\infty \kappa d\kappa e^{-i\kappa(m^2 x^2 - i\delta)} e^{-i(1-i\epsilon)/(4\kappa)}. \quad (16)$$

We thus note that the object $x^\mu \Delta'_F(x^2)$ is of odd parity, and, if we hold off taking $\delta \rightarrow 0$ in Eq. (16), it is even equal to zero when $x = 0$. Thus we are at least close to an argument that $x^\mu \Delta'_F(x^2) \delta^{(4)}(x)$ actually vanishes at $x = 0$, so that, with Eq. (15), we obtain $\partial_\nu L^{\mu\nu}(x) = 0$ everywhere. Of course, because our insertion must be electromagnetically gauge-invariant, we do need to require that $L^{\mu\nu}(x)$ indeed has such a vanishing four-divergence. We now wish to recast $L^{\mu\nu}(x)$ into a simple form whose vanishing four-divergence is manifest. Using the fact that $\partial^\mu \Delta_F(x^2) = 2x^\mu \Delta'_F(x^2)$, we first rewrite Eq. (13) as

$$\begin{aligned} L^{\mu\nu}(x) &= -16\pi i \left\{ x^\mu x^\nu \left(8(\Delta'_F(x^2))^2 \right) \right. \\ &\quad \left. - \eta^{\mu\nu} \left[x^2 8(\Delta'_F(x^2))^2 - \left(x^2 4(\Delta'_F(x^2))^2 + m^2 (\Delta_F(x^2))^2 \right) \right] \right\}. \end{aligned} \quad (17)$$

Now it may readily be demonstrated that a tensor of the form

$$A^{\mu\nu}(x) = x^\mu x^\nu f(x^2) - \eta^{\mu\nu} \left(x^2 f(x^2) + d(x^2) \right) \quad (18a)$$

has vanishing four-divergence if and only if

$$d'(x^2) = \frac{3}{2} f(x^2), \quad (18b)$$

that is, if

$$d(x^2) = \frac{3}{2} \int_{x_1^2}^{x^2} d\lambda f(\lambda). \quad (18c)$$

We thus see that the divergenceless nature of $A^{\mu\nu}(x)$ effectively means that it depends only on a single scalar function. Can we therefore make $A^{\mu\nu}(x)$ *manifestly* divergenceless by writing it in terms of a single scalar function? To this end we look at the manifestly divergenceless form:

$$B^{\mu\nu}(x) = (\partial^\mu \partial^\nu - \eta^{\mu\nu} \partial_\alpha \partial^\alpha) h(x^2), \quad (19a)$$

which yields

$$B^{\mu\nu}(x) = x^\mu x^\nu 4h''(x^2) - \eta^{\mu\nu} \left(x^2 4h''(x^2) + 6h'(x^2) \right). \quad (19b)$$

Comparison of Eq. (19b) with Eqs. (18) shows that $B^{\mu\nu}(x) = A^{\mu\nu}(x)$ if

$$h''(x^2) = \frac{1}{4} f(x^2)$$

and

$$h'(x^2) = \frac{1}{6} d(x^2) = \frac{1}{4} \int_{x_1^2}^{x^2} d\lambda f(\lambda),$$

which amount to the same thing. So we simply require that

$$h(x^2) = \frac{1}{4} \int_{x_0^2}^{x^2} d\lambda \int_{x_1^2}^{\lambda} d\lambda' f(\lambda'). \quad (20)$$

To reexpress Eq. (17) in the manifestly divergenceless format of Eq. (19a), the $f(x^2)$ we need is

$$f(x^2) = -128\pi i \left(\Delta'_F(x^2) \right)^2, \quad (21)$$

and, as we wish to arrange that $h(x^2)$ vanishes as $|x^2| \rightarrow \infty$ (this will later permit us to obtain $H(x^2)$ having the same property), we take both x_0^2 and x_1^2 in Eq. (20) to be $-\infty$. Using Eq. (16), we can express Eq. (21) in the form

$$f(x^2) = \frac{8i(m^2)^4}{\pi^3} \int_0^\infty d\kappa_1 \int_0^\infty d\kappa_2 \kappa_1 \kappa_2 e^{-i(\kappa_1 + \kappa_2)(m^2 x^2 - i\delta)} e^{-i(1-i\epsilon)(\kappa_1 + \kappa_2)/(4\kappa_1 \kappa_2)}. \quad (22)$$

It is now convenient to change variables in Eq. (22) to $\kappa \equiv \kappa_1 + \kappa_2$ and $z \equiv \kappa_1/(\kappa_1 + \kappa_2)$. Thus, $\kappa_1 = z\kappa$, $\kappa_2 = (1-z)\kappa$, and $d\kappa_1 d\kappa_2 = dz \kappa d\kappa$. Equation (22) becomes

$$f(x^2) = \frac{8i(m^2)^4}{\pi^3} \int_0^\infty d\kappa \kappa^3 e^{-i\kappa(m^2 x^2 - i\delta)} Z_\epsilon(\kappa), \quad (23a)$$

where

$$Z_\epsilon(\kappa) \equiv \int_0^1 dz z(1-z) e^{-i(1-i\epsilon)/(4\kappa z(1-z))}. \quad (23b)$$

Using the Taylor expansion and the stationary phase approximation, one readily obtains the asymptotic behaviours of $Z_\epsilon(\kappa)$:

$$Z_\epsilon(\kappa) \sim \frac{1}{6} \left(1 - \frac{3i}{2\kappa}\right) \quad \text{as } \kappa \rightarrow \infty, \quad (24a)$$

$$Z_\epsilon(\kappa) \sim \frac{e^{-i\pi/4}}{8} (\pi\kappa)^{\frac{1}{2}} e^{-i(1-i\epsilon)/\kappa} \quad \text{as } \kappa \rightarrow 0. \quad (24b)$$

While Eqs. (24) tell us that $Z_\epsilon(\kappa)$ approaches a constant as $\kappa \rightarrow \infty$, it has an oscillatory essential singularity as $\kappa \rightarrow 0$ that can effectively cut off singularities which other functions, integrated over κ together with $Z_\epsilon(\kappa)$, may have in this limit. Examples of such $\kappa \rightarrow 0$ singularities occur when we use Eqs. (20) and (23) to evaluate our $h(x^2)$. In particular, we have

$$\begin{aligned} \int_{-\infty}^{x^2} d\lambda e^{-i\kappa m^2 \lambda} &= \lim_{\Lambda \rightarrow +\infty} \int_{-\Lambda}^{x^2} d\lambda e^{-i\kappa m^2 \lambda} \\ &= \frac{1}{m^2} \left(\pi \delta(\kappa) + i e^{-i\kappa m^2 x^2} \mathcal{P} \frac{1}{\kappa} \right), \end{aligned} \quad (25a)$$

and

$$\lim_{x^2 \rightarrow \pm\infty} \left(i e^{-i\kappa m^2 x^2} \mathcal{P} \frac{1}{\kappa} \right) = \pm \pi \delta(\kappa). \quad (25b)$$

The presence of the oscillatory essential singularity in $Z_\epsilon(\kappa)$ as $\kappa \rightarrow 0$ always permits us to drop singular niceties such as the $\pi\delta(\kappa)$ and the principal-value notation in Eq. (25a) when we carry out this sort of integration under the integral sign in representations of the type given by Eq. (23a). Thus, for the desired $h(x^2)$ associated with the $f(x^2)$ of Eqs. (21) and (23), we obtain

$$\begin{aligned} h(x^2) &= \frac{1}{4} \int_{-\infty}^{x^2} d\lambda \int_{-\infty}^{\lambda} d\lambda' f(\lambda') \\ &= \frac{-2i(m^2)^2}{\pi^3} \int_0^\infty \kappa d\kappa e^{-i\kappa(m^2 x^2 - i\delta)} Z_\epsilon(\kappa). \end{aligned} \quad (26)$$

Equation (26) provides an explicit expression for the $h(x^2)$ which permits us to write the loop tensor $L^{\mu\nu}(x)$ of Eq. (17) in the manifestly divergenceless format of Eq. (19a)

$$L^{\mu\nu}(x) = (\partial^\mu \partial^\nu - \eta^{\mu\nu} \partial_\alpha \partial^\alpha) h(x^2). \quad (27)$$

Equation (27) is, of course, the space-time version of Eq. (3a). We also wish to explicitly obtain the virtual stress-energy tensor $T^{\mu\nu}(x)$ itself in the similar manifestly divergenceless form given by

$$T^{\mu\nu}(x) = e^2 (\partial^\mu \partial^\nu - \eta^{\mu\nu} \partial_\alpha \partial^\alpha) H(x^2), \quad (28a)$$

where, from Eq. (1a) and Eq. (27), we must have that

$$\partial_\alpha \partial^\alpha H(x^2) = h(x^2). \quad (28b)$$

We note here that since $H(x^2)$ is the four-dimensional Fourier transform of the finite function $\hat{H}(q^2)$ of Eq. (7), which is well-behaved as $q^2 \rightarrow 0$, we can infer that $H(x^2) \rightarrow 0$ as $|x^2| \rightarrow \infty$. To explicitly

obtain the stress-energy tensor (28a), we must solve the d'Alembertian relation of Eq. (28b) for the function $H(x^2)$, given our $h(x^2)$ of Eq. (26). Thus we must solve the equation

$$\partial_\alpha \partial^\alpha H(x^2) = 4 \left(\frac{d}{dx^2} \right)^2 \left(x^2 H(x^2) \right) = h(x^2),$$

for which we readily obtain a solution in accord with the requirement noted above that $H(x^2)$ vanish as $|x^2| \rightarrow \infty$,

$$H(x^2) = \frac{1}{4x^2} \int_{-\infty}^{x^2} d\lambda \int_{-\infty}^{\lambda} d\lambda' h(\lambda') \quad (29a)$$

$$= \frac{i}{2\pi^3 x^2} \int_0^\infty \frac{d\kappa}{\kappa} e^{-i\kappa(m^2 x^2 - i\delta)} Z_\epsilon(\kappa). \quad (29b)$$

Equation (29b) explicitly provides the desired $H(x^2)$ for the representation of $T^{\mu\nu}(x)$, which is given by Eq. (28a). These equations make it clear that $T^{\mu\nu}(x)$ has the dimensions of stress-energy as well as the stress-energy tensor properties of index symmetry and divergencelessness. $T^{\mu\nu}(x)$ is, however, a *virtual* stress-energy—its complex-valued nature follows from that of $H(x^2)$.

With the desired explicit representations of $h(x^2)$ and $H(x^2)$ in hand in Eqs. (26) and (29), it is useful to obtain their asymptotic behaviours.⁴ The results obtained for $h(x^2)$ are

$$h(x^2) \sim \frac{-m}{4\pi^2 (x^2)^{\frac{3}{2}}} e^{-2im\sqrt{x^2}} \quad \text{as } x^2 \rightarrow +\infty, \quad (30a)$$

$$h(x^2) \sim \frac{im}{4\pi^2 (-x^2)^{\frac{3}{2}}} e^{-2m\sqrt{-x^2}} \quad \text{as } x^2 \rightarrow -\infty, \quad (30b)$$

$$h(x^2) \sim \frac{i}{3\pi^3 (x^2 - i\delta)^2} \quad \text{as } x^2 \rightarrow 0, \quad (30c)$$

and those for $H(x^2)$ are:

$$H(x^2) \sim \frac{1}{16\pi^2 m (x^2)^{\frac{3}{2}}} e^{-2im\sqrt{x^2}} \quad \text{as } x^2 \rightarrow +\infty, \quad (31a)$$

$$H(x^2) \sim \frac{-i}{16\pi^2 m (-x^2)^{\frac{3}{2}}} e^{-2m\sqrt{-x^2}} \quad \text{as } x^2 \rightarrow -\infty, \quad (31b)$$

$$H(x^2) \sim \frac{-i \ln(m^2 x^2 - i\delta)}{12\pi^3 x^2} \quad \text{as } x^2 \rightarrow 0. \quad (31c)$$

Equation (30c) shows, as was pointed out in Section II, that the Fourier transform to four-momentum space of $h(x^2)$, namely $\hat{h}(q^2)$, diverges logarithmically. This is due to the non-integrable light-cone singularity (as $x^2 \rightarrow 0$). However, Eq. (31c) shows that the light-cone singularity of $H(x^2)$ is integrable over space-time, so that its Fourier transform $\hat{H}(q^2)$ is finite. This Fourier transform $\hat{H}(q^2)$ may readily be obtained by using our representation for $H(x^2)$ in Eq. (29b). The integration over configuration space-time becomes essentially a straightforward Gaussian integration exercise (it is helpful to first introduce an additional auxiliary integration, which raises the leading factor of $(x^2)^{-1}$ into the exponential as well—this auxiliary integral is then straightforwardly evaluated after

⁴Standard asymptotic techniques are applied here, including the stationary-phase approximation, the saddle-point approximation, and taking subexpression limits after changes of variable.

the Gaussian integrations over space–time have been carried out). Once the space–time integrations have been carried out, the κ -integration of Eq. (29b) may also readily be evaluated, provided one keeps the auxiliary integration over z which occurs in the definition of $Z_\epsilon(\kappa)$ —see Eq. (23b). One can as well carry out the final z -integration analytically, but the properties of the result are more transparent if one eschews this step in favour of keeping the z -integral form. Although $\hat{h}(q^2)$ diverges logarithmically, as we have noted, one can as well take the representation of Eq. (26) for $h(x^2)$ and at least begin the analogous procedure. The Gaussian space–time integrations are readily carried out, but the subsequent κ -integration is then seen to diverge logarithmically at its upper limit. At this stage, before the κ -integration is carried out, one can instead write down the representation for $(\hat{h}(q^2) - \hat{h}|_{q^2=0})$. For this object the κ -integration in fact converges—indeed one recognizes that the resulting expression is closely related to that obtained for $\hat{H}(q^2)$:

$$\left(\hat{h}(q^2) - \hat{h}|_{q^2=0} \right) = -q^2 \hat{H}(q^2) \quad (32a)$$

$$= -\frac{2}{\pi} \int_0^1 dz z(1-z) \ln \left(1 - \frac{q^2 z(1-z)}{m^2(1-i\epsilon)} \right). \quad (32b)$$

Equation (32a) checks with Eq. (7) of Section I, and Eq. (32b) checks against standard results for the “vacuum polarization” Feynman diagram [19] upon substitution into Eq. (5) or Eq. (6). Thus the Fourier transform exercise that we have sketched above in words validates the correctness of our representations of $h(x^2)$ and $H(x^2)$ given by Eqs. (26) and (29), and also the properties of these functions which follow from those representations (e.g. Eqs. (30) and (31)).

The basic second-order virtual stress-energy $T^{\mu\nu}(x)$ that is being discussed here, as well as versions of it which are corrected to higher order in the electromagnetic coupling strength e , may both be compactly characterized in the language of second-quantized fields,

$$T^{\mu\nu}(x) = \langle 0 | T(A^\mu(x) J^\nu(0)) | 0 \rangle \quad (33a)$$

$$= \int d^4 x' D_F((x-x')^2) \langle 0 | T(J^\mu(x') J^\nu(0)) | 0 \rangle, \quad (33b)$$

where $|0\rangle$ is the vacuum state, T is the Dyson time-ordering symbol, $D_F(x^2)$ is the Lorentz-gauge photon propagator (this object is equal to the zero-mass limit of $-\Delta_F(x^2)$ and satisfies $\partial_\alpha \partial^\alpha D_F(x^2) = \delta^{(4)}(x)$), $A^\mu(x)$ is, ideally, the second-quantized Heisenberg-picture electromagnetic vector potential in Lorentz gauge, and $J^\mu(x)$ is, ideally, its corresponding second-quantized Heisenberg-picture source current density. Although it isn’t known how to practically construct the full Heisenberg-picture source current density $J^\mu(x)$, we *can* at least suppose we have obtained it through some finite perturbative order n in e . Up to this point we have, of course, been dealing with the case $n = 1$, since the lowest possible order perturbative approximation to $J^\mu(x)$ in quantum electrodynamics is an object that is simply proportional to e .

Just as in the $n = 1$ case, it is clear from Eq. (33b) that $T^{\mu\nu}(x)$ is always simply related to its generalized “loop portion”, which is

$$L^{\mu\nu}(x) \equiv e^{-2} \langle 0 | T(J^\mu(x) J^\nu(0)) | 0 \rangle. \quad (34)$$

From Ref. [20] we learn that through use of spectral decomposition of the above expression for $L^{\mu\nu}(x)$ (i.e., insertion of a complete set of states, that all have definite four-momenta, between $J^\mu(x)$ and $J^\nu(0)$, followed by application onto these states of the space–time translation operator

which transforms $J^\mu(x)$ to $J^\mu(0)$), combined with the TCP transformation properties of the current density $J^\mu(x)$, it can be shown that $L^{\mu\nu}(x)$ is symmetric in its indices μ and ν . Conservation of the current density $J^\mu(x)$ implies that $L^{\mu\nu}(x)$ is also divergenceless. The upshot is that this generalized $L^{\mu\nu}(x)$ of Eq. (34) can be expressed in the same general form as we have in Eq. (27) for the $n = 1$ case:

$$L^{\mu\nu}(x) = (\partial^\mu \partial^\nu - \eta^{\mu\nu} \partial_\alpha \partial^\alpha) h(x^2). \quad (35)$$

From this and Eqs. (33b) and (34) above we obtain as well the generalization to higher order of the forms of Eqs. (28):

$$T^{\mu\nu}(x) = e^2 (\partial^\mu \partial^\nu - \eta^{\mu\nu} \partial_\alpha \partial^\alpha) H(x^2) \quad (36a)$$

where

$$\partial_\alpha \partial^\alpha H(x^2) = h(x^2). \quad (36b)$$

Unlike in the case of $n = 1$, we don't, of course, have explicit representations for the $h(x^2)$ and $H(x^2)$ that correspond to general order n , nor do we know the asymptotic behaviours of these functions. However, it is quite plausible that general theorems exist that can at least provide those asymptotic behaviours, which would likely be sufficient for obtaining the main features of the self-gravitational corrections of these higher order cases by means of the approach of the next Section. The self-gravitational correction of $T^{\mu\nu}(x)$ corrected to higher electromagnetic orders would therefore seem to be an interesting topic for future investigation.

It is worth pointing out here that the form given for $T^{\mu\nu}(x)$ in Eq. (33a), notwithstanding its delocalized and complex-valued “virtual quantum” nature, also bears a strong resemblance to the Lagrangian density's electromagnetic *interaction* term that contains the coupling of the vector potential to the current density. In view of the vacuum expectation value that also is a feature of Eq. (33a), it would thus seem appropriate to refer to $T^{\mu\nu}(x)$ as the vacuum virtual electromagnetic interaction stress-energy.

IV The self-gravitationally corrected virtual stress-energy tensor

The vacuum virtual electromagnetic interaction stress-energy tensor $T^{\mu\nu}(x)$ for $n = 1$ may be expected, because of its non-integrable singularity on the light cone, to give rise to *ultrastrong* virtual self-gravitational effects. Under such circumstances, approximate treatments of the propagation of resulting intermediate virtual gravitons which ignore the non-linear feature of their having interactions *with each other* can only be regarded with great misgiving. Thus this propagation will be effected here by means of the full Einstein equation, so that it is described in terms of a virtual metric. A notable by-product of this approach is that subsequent approximate perturbative treatment of these non-linear graviton-graviton interactions is thus avoided, which eliminates the source of the ultraviolet divergences which occur in the usual approaches to second-quantized gravity theory itself. As the virtual gravitational source $T^{\mu\nu}(x)$ is dependent on just $H(x^2)$, a single function of only the Lorentz invariant variable $x^2 \equiv \eta_{\mu\nu} x^\mu x^\nu$, one can expect the corresponding virtual metric to have an analogously high degree of symmetry. Thus a conformally flat virtual metric ansatz whose departure from flatness depends on a single function of x^2 is the natural one to try:

$$g_{\mu\nu} = [B(x^2)]^{-2} \eta_{\mu\nu}. \quad (37)$$

It is now straightforward, if somewhat tedious, to calculate the virtual Einstein tensor for this $g_{\mu\nu}$ (the sign conventions used here for curvature tensors are those of Weinberg [21]):

$$G^{\mu\nu} = -8B^3 B'' x^\mu x^\nu - g^{\mu\nu} \left[-8B^3 B'' x^2 B^{-2} + 12BB' (BB' x^2 B^{-2} - 1) \right], \quad (38)$$

where the $x^\mu \equiv \eta^{\mu\nu}(\partial x^2/\partial x^\nu)/2$ are a direct generalization of the Cartesian coordinates of Minkowski space-time to our conformally flat case (recall that $x^2 \equiv \eta_{\mu\nu} x^\mu x^\nu$). The three manifestly invariant objects $g_{\mu\nu} G^{\mu\nu}$, $G_{\mu\nu} G^{\mu\nu}$, and $\det(G_{\mu\nu})/\det(g_{\mu\nu})$ which can be formed from $G^{\mu\nu}$ and $g_{\mu\nu}$ may be worked out in terms of x^2 , B , B' , and B'' . From these three expressions it can be concluded that the three algebraically simpler objects BB' , $x^2 B^{-2}$, and $B^3 B''$ are also invariants (but we note that neither B nor x^2 themselves are invariants). Taking note of these three simple invariants and of the contravariant tensor character of our $G^{\mu\nu}$, whose form as given in Eq. (38) involves these invariants as coefficients, we conclude that $x^\mu x^\nu$ is also a contravariant tensor (whose contraction $g_{\mu\nu} x^\mu x^\nu$ evaluates to one of our three simple invariants, namely $x^2 B^{-2}$).

The flat-space virtual stress-energy tensor (28a) now needs to be generalized to the curved space-time geometry specified above. The standard procedure is the “minimal coupling” prescription, according to which one must replace partial derivatives with covariant derivatives, and the Minkowski metric with the curved space-time metric. In the present case, however, the covariant divergence of the minimal coupling expression does not vanish, but is proportional to the four-gradient of H contracted with the Ricci curvature tensor. Thus the minimal coupling recipe cannot be applied to the Fourier transform of the “vacuum polarization” diagram.⁵ Nonetheless, it is not difficult to construct a stress-energy tensor in the above curved space-time which is symmetric and covariantly divergenceless, and which, for $B(x^2) \rightarrow 1$, reduces to the flat space-time tensor (28a). We observe that this latter object can be rewritten in the form

$$T^{\mu\nu}(x) = 4e^2 H''(x^2) x^\mu x^\nu - \eta^{\mu\nu} \left[4e^2 H''(x^2) x^2 + 6e^2 H'(x^2) \right]. \quad (39)$$

One notes that the metric ansatz (37) has produced an Einstein tensor (38) having a marked structural similarity to this flat space-time stress-energy tensor (39). With a view toward making the needed modifications to this flat-space $T^{\mu\nu}(x)$ (these modifications must, of course, disappear when $B(x^2)$ is put to unity, i.e., when we go to the flat-space limit), the conditions under which a general tensor having this shared structural form, namely

$$S^{\mu\nu}(x) \equiv F(x^2) x^\mu x^\nu - g^{\mu\nu} \left[F(x^2) x^2 [B(x^2)]^{-2} + D(x^2) \right], \quad (40)$$

is covariantly divergenceless are now examined. This is essentially the covariant generalization (in the context of our particular “conformally flat” metric) of the procedure in Eq. (18). Taking the covariant divergence of $S^{\mu\nu}$ yields

$$S^{\mu\nu}{}_{;\nu} = \left[3F - 6FBB' x^2 B^{-2} - 2D'B^2 \right] x^\mu. \quad (41)$$

For the vanishing of the covariant divergence, it is required that

$$D' = \frac{3}{2} F B^{-2} \left[1 - 2BB' x^2 B^{-2} \right] \quad (42a)$$

⁵The physical reason for the inapplicability of the minimal-coupling prescription in this instance is that our $T^{\mu\nu}$ describes a (virtual) *extended* system, as is apparent from the presence of the virtual electron-positron loop in the Feynman diagram, as well as the explicitly delocalized character of the representations of $T^{\mu\nu}(x)$ in Eqs. (33). Such extended systems are subject to *tidal* gravitational forces, whose proper inclusion is not encompassed by the minimal coupling framework.

or

$$D(x^2) = \frac{3}{2} \int_{x_1^2}^{x^2} d\lambda [B(\lambda)]^{-2} F(\lambda) \left[1 - 2\lambda \frac{B'(\lambda)}{B(\lambda)} \right]. \quad (42b)$$

In the flat-space limit where $B(x^2)$ is taken to unity, we can readily see that Eqs. (40) and (42) reduce to analogues of the three parts of Eq. (18). We may also readily verify that our Einstein tensor $G^{\mu\nu}$ of Eq. (38) is a tensor of the form (40), which satisfies Eq. (42b) with lower limit $x_1^2 = -\infty$, because far from the light cone (i.e. as $|x^2| \rightarrow \infty$), $B(x^2) \rightarrow 1$.

Having established the condition (42) for a tensor of the form (40) to be covariantly divergenceless, one still must note that the conversion of a purely special-relativistic contravariant stress-energy tensor, such as $T^{\mu\nu}(x)$ of Eq. (39) above, into a contravariant stress-energy tensor suitable for placement on the right-hand side of the Einstein equation, can involve, among other things, its multiplication by the factor $g^{-\frac{1}{2}}$, where $g \equiv -\det(g_{\mu\nu})$ —see the example given in Ref. [22]. Multiplication of $T^{\mu\nu}(x)$ by $g^{-\frac{1}{2}}$ ensures invariance of the contravariant Einstein equation under uniform rescaling by a constant factor of the space and time inertial coordinates of the local freely-falling reference frames,⁶ while retaining the proper special-relativistic limiting form for the resultant tensor as space-time becomes flat. For the metric ansatz of Eq. (37), $g^{-\frac{1}{2}} = B^4$, so the result of multiplying the $T^{\mu\nu}(x)$ of Eq. (39) by $g^{-\frac{1}{2}}$ is a tensor of the form given in Eq. (40), with $F(x^2) = 4e^2 H''(x^2) [B(x^2)]^4$ and $D(x^2) = 6e^2 H'(x^2) [B(x^2)]^2$. To make this tensor covariantly divergenceless, however, $D(x^2)$ must still be modified to accord with the prescription of Eq. (42b). Thus, having taken account of inertial rescaling characteristics, the choice for the covariantly divergenceless stress-energy tensor $T_B^{\mu\nu}(x)$ is

$$T_B^{\mu\nu}(x) = F(x^2) x^\mu x^\nu - g^{\mu\nu} \left[F(x^2) x^2 [B(x^2)]^{-2} + D(x^2) \right], \quad (43a)$$

where

$$F(x^2) \equiv [B(x^2)]^4 4e^2 H''(x^2), \quad (43b)$$

and

$$D(x^2) \equiv \frac{3}{2} \int_{-\infty}^{x^2} d\lambda [B(\lambda)]^{-2} F(\lambda) \left[1 - 2\lambda \frac{B'(\lambda)}{B(\lambda)} \right] \quad (43c)$$

$$= 6e^2 \int_{-\infty}^{x^2} d\lambda [B(\lambda)]^2 H''(\lambda) \left[1 - 2\lambda \frac{B'(\lambda)}{B(\lambda)} \right]. \quad (43d)$$

The $T_B^{\mu\nu}(x)$ of Eq. (43) has inertial rescaling weight -4 (see the preceding footnote), is symmetric and covariantly divergenceless, and clearly reduces to the flat-space $T^{\mu\nu}(x)$ of Eq. (39) (and Eq. (28a)) when we put $B(x^2)$ to unity. It is also clear that $T_B^{\mu\nu}(x)$ is entirely determined by the

⁶The inertial space and time coordinates of the local freely-falling reference frames typically enter implicitly into general-relativistic expressions via the covariant metric tensor $g_{\mu\nu}$, which depends on them in a bilinear, Lorentz-invariant manner [23]. While the affine connection (the “gravitational force”) and the *covariant* Ricci and Einstein tensors are, in fact, invariant under uniform rescaling of these inertial space and time coordinates by a constant factor, the metric objects $g_{\mu\nu}$, $g^{\mu\nu}$, and g respectively have weights of $+2$, -2 , and $+8$ for such inertial rescaling. Thus the *contravariant* Einstein tensor $G^{\mu\nu}$ has inertial rescaling weight -4 , while the special-relativistic form for $T^{\mu\nu}(x)$ given by Eq. (39) has inertial rescaling weight zero, as it obviously has no dependence on $g_{\mu\nu}$. Multiplication of this special-relativistic $T^{\mu\nu}(x)$ by the factor $g^{-\frac{1}{2}}$ gives the resultant object inertial rescaling weight -4 , which matches that of $G^{\mu\nu}$.

function $F(x^2)$ given in (43b) above, just as the Einstein tensor $G^{\mu\nu}$ is determined in the same manner by its “ $F(x^2)$ part”, namely $-8B^3B''$. So to solve the contravariant Einstein equation

$$G^{\mu\nu} = -8\pi G T_B^{\mu\nu}, \quad (44)$$

it is thus sufficient to equate these “ $F(x^2)$ parts”:

$$-8[B(x^2)]^3 B''(x^2) = -8\pi G [B(x^2)]^4 4e^2 H''(x^2). \quad (45)$$

Thus one needs to solve an ordinary second-order linear homogeneous differential equation of standard form:

$$B''(x^2) = 4\pi G e^2 H''(x^2) B(x^2). \quad (46)$$

Let us now formally express its solution as the infinite perturbation expansion series

$$B(x^2) = \sum_{n=0}^{\infty} (4\pi G e^2)^n b^{(n)}(x^2), \quad (47)$$

which we substitute into Eq. (46). Using the boundary condition that $B(x^2) \rightarrow 1$ as $|x^2| \rightarrow \infty$, we obtain that $b^{(0)}(x^2) = 1$ and

$$b^{(n)}(x^2) = \int_{-\infty}^{x^2} d\lambda \int_{-\infty}^{\lambda} d\lambda' H''(\lambda') b^{(n-1)}(\lambda') \quad \text{for } n = 1, 2, 3, \dots \quad (48)$$

In particular, for $n = 1$ we obtain $b^{(1)}(x^2) = H(x^2)$. For orders higher than the first, we are unable to obtain exact results for the $b^{(n)}(x^2)$. However, so long as $|4\pi G e^2 H(x^2)| \ll 1$, we should be able to rely on the results which we have obtained through first order. The asymptotic behaviour of $H(x^2)$ given in Eq. (31c) thus tells us that the first-order perturbative results are adequate in the region $|x^2| \gg G e^2$. On the other hand, in the region $|x^2| \ll 1/m^2$, where Eq. (31c) is the asymptotically valid representation of $H(x^2)$, that relationship turns out to actually permit the determination of the asymptotic forms in this region of *all* of the $b^{(n)}(x^2)$. Bearing in mind our results through first order, we make the following asymptotic ansatz:

$$b^{(n-1)}(x^2) \sim c_{n-1} (H(x^2))^{n-1} \sim c_{n-1} \left(-iK \frac{\ln(m^2 x^2)}{x^2} \right)^{n-1} \quad \text{for } |x^2| \ll 1/m^2, \quad (49)$$

where, of course, $c_0 = 1$ and, from Eq. (31c), $K = 1/(12\pi^3)$. Since we readily see that

$$H''(x^2) \sim -2iK \frac{\ln(m^2 x^2)}{(x^2)^3} \quad \text{for } |x^2| \ll 1/m^2, \quad (50)$$

we calculate from Eq. (48) that

$$b^{(n)}(x^2) \sim \frac{2c_{n-1}}{n(n+1)} \left(-iK \frac{\ln(m^2 x^2)}{x^2} \right)^n \sim \frac{2c_{n-1}}{n(n+1)} (H(x^2))^n \quad \text{for } |x^2| \ll 1/m^2. \quad (51)$$

Thus we have obtained the recurrence relation for the coefficients c_n of our ansatz:

$$c_n = \frac{2c_{n-1}}{n(n+1)}, \quad (52)$$

which, for our known zeroth-order $c_0 = 1$, properly yields our known first-order $c_1 = 1$, and may readily be solved for all orders:

$$c_n = \frac{2^n}{n!(n+1)!} \quad \text{for } n = 0, 1, 2, \dots \quad (53)$$

Putting this result and our ansatz of Eq. (49) into the perturbative expansion series of Eq. (47), we obtain

$$B(x^2) \sim \sum_{n=0}^{\infty} \frac{(8\pi G e^2 H(x^2))^n}{n!(n+1)!} = \frac{I_1(2\sqrt{8\pi G e^2 H(x^2)})}{\sqrt{8\pi G e^2 H(x^2)}} \quad \text{for } |x^2| \ll 1/m^2, \quad (54)$$

where we have been able to actually sum, in terms of the modified Bessel function I_1 , the perturbation expansion in the region $|x^2| \ll 1/m^2$. Eq. (54) clearly incorporates the unadulterated perturbative result through first order, which, as we have discussed above, ensures this equation's accuracy so long as $|x^2| \gg G e^2$. In view of this equation's validity for $|x^2| \ll 1/m^2$ *as well*, plus the fact that $G e^2 m^2 \approx 1.3 \times 10^{-47}$, we conclude that

$$B(x^2) \approx \frac{I_1(2\sqrt{8\pi G e^2 H(x^2)})}{\sqrt{8\pi G e^2 H(x^2)}} \quad \text{uniformly in } x^2. \quad (55)$$

The complex nature of $H(x^2)$ of course causes $B(x^2)$ to be complex as well—our metric is a virtual one, as was mentioned at the beginning of this Section.

The desired virtual metric $g_{\mu\nu} = [B(x^2)]^{-2} \eta_{\mu\nu}$ is now in hand from the result in Eq. (55). From this virtual metric we obtain the flat-space (Minkowskian) virtual gravitational field tensor $h_{\mu\nu}$:

$$h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu} = ([B(x^2)]^{-2} - 1) \eta_{\mu\nu}. \quad (56)$$

For our virtual Einstein tensor $G^{\mu\nu}$ of Eq. (38), the portion that is linear in $h_{\mu\nu}$ is given by

$$G^{(1)\mu\nu} = x^\mu x^\nu (4(B^{-2} - 1)') - \eta^{\mu\nu} (4x^2(B^{-2} - 1)'' + 6(B^{-2} - 1)') \quad (57a)$$

$$= (\partial^\mu \partial^\nu - \eta^{\mu\nu} \partial_\alpha \partial^\alpha) ([B(x^2)]^{-2} - 1). \quad (57b)$$

Of course, $G^{(1)\mu\nu}$ is divergenceless in the ordinary flat-space (Minkowskian) sense, so it is customary to define the flat-space virtual stress-energy tensor which *includes* the virtual gravitational effects as [14]:

$$\tau^{\mu\nu} \equiv -\frac{1}{8\pi G} G^{(1)\mu\nu}. \quad (58)$$

Thus $\tau^{\mu\nu}$ is the self-gravitationally corrected flat-space virtual stress-energy tensor $T_G^{\mu\nu}(x)$ which properly replaces our “ $G = 0$ ” $T^{\mu\nu}(x)$ of Eqs. (28a) and (39). Using Eqs. (57b) and (58), we write it in the form

$$T_G^{\mu\nu}(x) = e^2 (\partial^\mu \partial^\nu - \eta^{\mu\nu} \partial_\alpha \partial^\alpha) \left(-\frac{1}{8\pi G e^2} \right) ([B(x^2)]^{-2} - 1), \quad (59a)$$

$$= e^2 (\partial^\mu \partial^\nu - \eta^{\mu\nu} \partial_\alpha \partial^\alpha) H_G(x^2), \quad (59b)$$

where

$$H_G(x^2) \equiv -\frac{1}{8\pi G e^2} ([B(x^2)]^{-2} - 1). \quad (59c)$$

If we further define

$$h_G(x^2) \equiv \partial_\alpha \partial^\alpha H_G(x^2), \quad (60)$$

we have in hand all the ingredients needed for the discussion of the self-gravitationally corrected version of the Feynman diagram of Fig. 1.

V Discussion

For the self-gravitationally corrected case, we still have relations analogous to those of Eqs. (28):

$$T_G^{\mu\nu}(x) = e^2 (\partial^\mu \partial^\nu - \eta^{\mu\nu} \partial_\alpha \partial^\alpha) H_G(x^2) \quad (61a)$$

and

$$\partial_\alpha \partial^\alpha H_G(x^2) = h_G(x^2), \quad (61b)$$

where, as shown in Section IV, $H_G(x^2)$ is simply related to a dimensionless virtual metric function $B(x^2)$:

$$H_G(x^2) = -\frac{1}{8\pi G e^2} ([B(x^2)]^{-2} - 1), \quad (62a)$$

and this virtual metric function $B(x^2)$ is, in turn, related to $H(x^2)$:

$$B(x^2) \approx \frac{I_1(2\sqrt{8\pi G e^2 H(x^2)})}{\sqrt{8\pi G e^2 H(x^2)}}. \quad (62b)$$

Note that our virtual metric function $B(x^2)$ tends toward unity far from the light cone ($|x^2| \rightarrow \infty$), as we can see from Eqs. (31a) and (31b) that $H(x^2)$ vanishes in that limit. Equations (61b) and (62a) also yield

$$h_G(x^2) = -\frac{\partial_\alpha \partial^\alpha [B(x^2)]^{-2}}{8\pi G e^2}. \quad (63)$$

From Eqs. (62) and (63), we can deduce that to zeroth order in G , $H_G(x^2)$ and $h_G(x^2)$ reduce to $H(x^2)$ and $h(x^2)$ respectively (and thus, from Eq. (61a), $T_G^{\mu\nu}(x)$ reduces to $T^{\mu\nu}(x)$), as must be true for physical consistency. However, very near to the light cone, the objects $H_G(x^2)$, $h_G(x^2)$, and $T_G^{\mu\nu}(x)$ behave very differently from their counterparts $H(x^2)$, $h(x^2)$, and $T^{\mu\nu}(x)$ respectively—which do not take account of the self-gravitational correction. We know from Eq. (31c), that $H(x^2) \sim -iK \ln(m^2 x^2)/x^2$ for $|x^2| \ll 1/m^2$. Thus, when $|x^2| \lesssim G e^2$, $|8\pi G e^2 H(x^2)| \gg 1$, and we may replace the modified Bessel function I_1 of Eq. (62b) by its large-argument asymptotic form:

$$B(x^2) \sim \left(\frac{1}{4\pi}\right)^{\frac{1}{2}} \frac{\exp(2\sqrt{8\pi G e^2 H(x^2)})}{[8\pi G e^2 H(x^2)]^{\frac{3}{4}}} \quad \text{for } |x^2| \lesssim G e^2. \quad (64a)$$

The above asymptotic form is in essential agreement with the result of applying the WKB approximation directly to our original differential equation (46) in this region. In conjunction with Eq. (63), it gives the asymptotic light cone behaviour for $h_G(x^2)$:

$$h_G(x^2) \sim h(x^2) 4 \ln(m^2 x^2) [B(x^2)]^{-2} \quad \text{for } |x^2| \lesssim G e^2. \quad (64b)$$

The asymptotic light-cone behaviour of $h(x^2)$ as given by Eq. (30c), namely $h(x^2) \sim 4iK(x^2)^{-2}$ as $x^2 \rightarrow 0$, is non-integrably singular, which is a far cry from that of $h_G(x^2)$ as given above, with its factor of $[B(x^2)]^{-2}$. The essential singularity on the light cone, which Eq. (64a) reveals in the behaviour of $B(x^2)$, plays the critical rôle in Eq. (64b), with its factor of $[B(x^2)]^{-2}$, of driving $h_G(x^2)$ very decisively to zero as $x^2 \rightarrow 0$. Thus, unlike the logarithmically divergent $\hat{h}(q^2)$, $\hat{h}_G(q^2)$ is fully convergent and finite.

From Eqs. (61) we can show that the self-gravitationally corrected objects $\hat{T}_G^{\mu\nu}(q)$ and $\hat{h}_G(q^2)$ satisfy the analogue of Eq. (3b),

$$\hat{T}_G^{\mu\nu}(q) = e^2 \left(\eta^{\mu\nu} q^2 - q^\mu q^\nu \right) \left(-\frac{\hat{h}_G(q^2)}{q^2 + i\epsilon} \right). \quad (65)$$

We observe that $\hat{T}_G^{\mu\nu}(q)$ is not, strictly speaking, second-order in e , since $\hat{h}_G(q^2)$ depends on *all* orders of Ge^2 . This is in accord with Haag's Theorem, which maintains that there can exist no strictly perturbative development in powers of e of quantum electrodynamics. Eq. (65) may be reexpressed in a form analogous to Eq. (5b),

$$\hat{T}_G^{\mu\nu}(q) = -\frac{e^2}{q^2} \left(\eta^{\mu\nu} q^2 - q^\mu q^\nu \right) \left(\hat{h}_G \Big|_{q^2=0} + \left(\hat{h}_G(q^2) - \hat{h}_G \Big|_{q^2=0} \right) \right). \quad (66)$$

In Eq. (66), however, $\hat{h}_G|_{q^2=0}$ is not logarithmically divergent, but fully finite, and it satisfies the well-defined integral relationship

$$\hat{h}_G \Big|_{q^2=0} = \int d^4x h_G(x^2). \quad (67)$$

Putting Eq. (61b) into Eq. (67), we arrive at

$$\hat{h}_G \Big|_{q^2=0} = \int d^4x \left(\partial_\alpha \partial^\alpha H_G(x^2) \right), \quad (68)$$

where the integral on the right-hand side is finite and well-defined. As it is the well-defined integral of a perfect differential, it must vanish (provided, of course, that the large $|x^2|$ asymptotic behaviour is appropriate—it is clear from Eq. (62) that the large $|x^2|$ asymptotic behaviour of $H_G(x^2)$ is the same as that of $H(x^2)$, which, as we see from Eqs. (31a) and (31b), tends toward zero for large $|x^2|$).

The vanishing of $\hat{h}_G|_{q^2=0}$ means that this insertion $\hat{T}_G^{\mu\nu}(q)$, once gravitationally corrected, does *not* contribute to charge renormalization *at all*, removing the paradox of an ultraviolet-dominated (not to mention infinite) object influencing the scale of the extreme low-energy scattering strength. Thus, in the self-gravitationally corrected case, the name “vacuum polarization” for this diagram is seen to be a serious misnomer. In fact, the nature of the diagram as represented in Fig. 1 makes it very clear that the production of the virtual pair is dependent *only* on the presence of the virtual photon—indeed, that virtual photon *dissociates* into the virtual pair, which subsequently recombines. Arising entirely from the virtual photon, the transient existence of this *single* virtual pair could easily be *far removed in space-time from any other charged particles*, so it can hardly be expected to systematically shield (renormalize) their charges. Of course, if the probability to make such a virtual pair somehow were *divergent* throughout space-time, we could no doubt have overwhelming shielding of other charges—that is presumably the diseased state of affairs before the self-gravitational correction of $T^{\mu\nu}(x)$ is taken into account. In view of the “vacuum polarization” diagram's reasonable lack of any contribution whatever to charge renormalization (once self-gravitationally corrected), the “vacuum polarization” misnomer accorded it needs to be discarded and replaced by a more physically accurate shorthand nomenclature—in view of our discussion at the end of Section III, “vacuum stress-energy” would seem to be an appropriate replacement prefix.

If, for a moment, $\hat{h}_G(q^2)$ is regarded to be merely a particular ultraviolet cutoff of the logarithmically divergent $\hat{h}(q^2)$, with G playing the rôle of cutoff parameter, it can be said that a cutoff technique has been found for which the diagram's contribution to charge renormalization vanishes identically over the entire range of values for the cutoff parameter except the non-cutoff limit value ($G = 0$ in this instance). Having thus encountered a cutoff method which has this property, it is clear that one can now readily construct many others. Given the existence of myriad cutoff techniques which yield zero charge renormalization contribution over their full range of cutoff parameter values (aside from the non-cutoff limit value), it becomes obvious that the “need” which this “vacuum polarization” diagram poses to effect charge renormalization is, in fact, a chimera.

Until one gets extremely close to the light cone ($|x^2| \approx Ge^2$), $h(x^2)$ remains an excellent approximation to $h_G(x^2)$. In particular, for $1/m^2 \gg |x^2| \gg Ge^2$, $h_G(x^2) \approx 4iK(x^2)^{-2}$. Thus, for the “normal” high-momentum-transfer range, $m^2 \ll |q^2| \ll (Ge^2)^{-1}$, we can expect $\hat{h}_G(q^2)$ in Eq. (65) to behave logarithmically in q^2 , $\hat{h}_G(q^2) \sim C_1 \ln(|q^2|/m^2)$, the familiar logarithmic growth with momentum transfer of the effective coupling at high momentum transfer [24]. However, once we reach the ultrahigh momentum transfer regime, $|q^2| \gtrsim (Ge^2)^{-1}$, Eqs. (64) for the asymptotic behaviour of $h_G(x^2)$ very close to the light cone imply that

$$|\hat{h}_G(q^2)| \sim C_2 [\ln(|q^2|/m^2)]^{\frac{5}{2}} [Ge^2|q^2|]^{\frac{3}{2}} \exp(-8\sqrt{K\pi Ge^2|q^2| \ln(|q^2|/m^2)}). \quad (69)$$

Thus the logarithmic growth of effective coupling with momentum transfer, after reaching a peak value of approximately $e^2(1 + (e^2/(3\pi)) \ln((Ge^2m^2)^{-1}))$ at $|q^2| \approx (Ge^2)^{-1}$, ultimately proceeds to abruptly collapse back to e^2 for larger $|q^2|$. Self-gravitational effects enforce ultimate “asymptotic freedom” even in quantum electrodynamics! Of course these considerations concerning such a “self-gravitational form factor” in quantum electrodynamics are of purely academic interest—momentum transfers of order $(Ge^2)^{-1}$ are utterly unattainable. In that vein, however, it is nonetheless fascinating that the *peak* effective coupling strength of around $e^2(1 + (e^2/(3\pi)) \ln((Ge^2m^2)^{-1}))$ is still very much in the vicinity of e^2 . The portentous “logarithmic rise in effective coupling strength” never can amount to a great deal, thanks to the limiting effect of self-gravitation. This example suggests in particular that the idea that the effective electromagnetic coupling strength may become equal to that of the strong interactions at sufficiently high energy may not be a reasonable one.

The ingredients needed for the evaluation of the self-gravitationally corrected version of the quartically divergent second-order electromagnetic vacuum-to-vacuum amplitude correction Feynman diagram shown in Fig. 2 are also in hand. This diagram only differs from that of Fig. 1 in that the single photon propagator now has each of its two ends attached respectively to one of the two vertices of the virtual electron–positron loop. Indeed, the value of the self-gravitationally corrected version of this diagram is simply

$$\int \frac{d^4q}{(2\pi)^4} \eta_{\mu\nu} \hat{T}_G^{\mu\nu}(q) = -3e^2 \int \frac{d^4q}{(2\pi)^4} \hat{h}_G(q^2) = -3e^2 h_G|_{x^2=0} = 0, \quad (70)$$

where we have made use of Eq. (65), of the fact that $h_G(x^2)$ is the four-dimensional Fourier transform of $\hat{h}_G(q^2)$, and of the vanishing of $h_G|_{x^2=0}$ which follows from Eqs. (64). Without the self-gravitational correction, of course $h(x^2) \sim 4iK(x^2)^{-2}$ as $x^2 \rightarrow 0$, which accords with this diagram's quartic divergence. As we have mentioned in Section I, the vanishing of this diagram upon its self-gravitational correction obviates the need for the usual fiat injunction that such a “disconnected” diagram is to be “discarded”, notwithstanding its strongly infinite value.

The “gravity-modified” result of Ref. [12] for the “vacuum polarization” diagram differs from that obtained here—the Ref. [12] result does not deviate from the usual quantum electrodynamics

result aside from having a finite charge renormalization (at least this is so if an accompanying gauge-non-invariant term that emerges is disregarded). The dominant $\ln(|q^2|/m^2)$ behaviour of the Ref. [12] result as $|q^2| \rightarrow \infty$ implies that there can be *no amelioration* of the quartic ultraviolet divergence of the closely related second-order vacuum-to-vacuum amplitude correction diagram—in stark contrast to the vanishing result obtained in Eq. (70) above for that diagram. The “graviton superpropagator” approximation of Ref. [12] also produces gauge invariance difficulties and is not of Haag’s Theorem character.

Finally, it must be emphasized that although the careful propagation of gravitons with the full Einstein equation that has been pursued in this paper is technically challenging, the usual approximations which split off the non-linear interactions of gravitons *with each other* to be subsequently treated *perturbatively* are disastrously inappropriate in the extreme ultraviolet regime, where those interactions in fact dominate the gravitational physics and produce “black-hole-like” damping phenomena. The *perturbative* mistreatment in the extreme ultraviolet of the there *dominant* non-linear terms produces physically and mathematically nonsensical results, namely the deluge of ultraviolet divergences which render what is usually termed “quantized gravity theory” unrenormalizable. The actual physical character of gravitation in the extreme ultraviolet clearly lies at the opposite pole from these perturbative artifacts—gravitation manifests an overwhelming tendency to damp, not diverge, in that limit, as its “black-hole-like” aspects come into their own. The usual intractable ultraviolet divergences of quantized gravity theory are thus clearly seen to have nothing to do with the actual physics, but everything to do with unthinking utilization of an extraordinarily ill-suited perturbation approach.

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References

- [1] L. D. Landau, in *Niels Bohr and the Development of Modern Physics*, ed. W. Pauli (Pergamon Press, London, 1955), p. 60.
- [2] O. Klein, *ibid.*, p. 96.
- [3] O. Klein, *Suppl. Helv. Phys. Acta* **4**, 61 (1956).
- [4] O. Klein, *Nuovo Cimento Suppl.* **6**, 344 (1957).
- [5] W. Pauli, *Suppl. Helv. Phys. Acta* **4**, 69 (1956).
- [6] S. Deser, *Revs. Mod. Phys.* **29**, 417 (1957).
- [7] R. Arnowitt, S. Deser, and C. W. Misner, *Phys. Rev.* **120**, 313 (1960).
- [8] S. K. Kauffmann, Preprint hep-th/9409171 (1994).
- [9] B. S. DeWitt, *Phys. Rev. Lett.* **13**, 114 (1964).
- [10] I. B. Khriplovich, *Sov. J. Nucl. Phys.* **3**, 415 (1966).
- [11] A. Salam and J. Strathdee, *Lett. Nuovo Cimento* **4**, 101 (1970).
- [12] C. J. Isham, A. Salam, and J. Strathdee, *Phys. Rev. D* **3**, 1805 (1971).
- [13] Khriplovich, *op. cit.*, p. 417.
- [14] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (John Wiley & Sons, New York, 1972), pp. 165–171.
- [15] M. Fabbrichesi, R. Pettorino, G. Veneziano, and G. A. Vilkovisky, *Nucl. Phys. B* **419**, 147 (1994).
- [16] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964), p. 153.
- [17] Bjorken and Drell, *op. cit.*, pp. 155–156.
- [18] Bjorken and Drell, *op. cit.*, p. 284.
- [19] Bjorken and Drell, *op. cit.*, p. 157.
- [20] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965), pp. 166–167.
- [21] Weinberg, *op. cit.*, pp. 131–142.
- [22] Weinberg, *op. cit.*, p. 126.
- [23] Weinberg, *op. cit.*, pp. 70–77.
- [24] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964), p. 159.

Figure captions

Figure 1: The second-order “vacuum polarization” radiative correction insertion Feynman diagram.

Figure 2: The second-order vacuum-to-vacuum amplitude correction Feynman diagram.

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